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Design of heterogeneous clamped circular plates with specified fundamental natural frequency

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ABSTRACT

The titled problem is solved by the aid of the semi-inverse formulation. At the first glance, a surprising demand is posed: Design a radially heterogeneous polar orthotropic circular plate whose fundamental mode shape coalesces with the static deflection of a homogeneous circular plate of the same radius and thickness under uniformly distributed load. Compatible polynomial variations of the radial and circumferential flexural rigidities are introduced. It turns out that these are infinite amount of polar orthotropic circular plates that solve the posed semi-inverse problem, depending on the free parameter represented by one of the coefficients in variation of flexural rigidities. This allows to provide an unique solution to the titled problem: there is a single plate that possesses the specified fundamental natural frequency.

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1. Introduction

Several papers are devoted to the free vibration of homogeneous orthotropic circular plates. Method of Frobenius was utilized by Akasaka and Takagishi (1958), Borsuk (1960) and Minkarah and Hoppman (1964) formulating the solution in terms of the hypergeometric functions. Power series expansion methodology was utilized by Pandalai and Patel (1965). The Galerkin method was applied by Prathap and Varadan (1976), while the Rayleigh–Ritz method was employed by Elishakoff (1987). Undetermined non-integer power method was used by Grossi et al. (1986). In above papers both the thickness and modulus of elasticity in radial and circumferential directions were fixed at constant values.

The circular polar orthotropic plates with variable properties were addressed in several investigations. Gupta and Lal (1990) studied polar orthotropic radially tapered circular plates.

In this study we deal with heterogeneous circular polar orthotropic plates. As Leissa (1981) mentions “a plate is heterogeneous if its material properties vary from point to point within it.” The candidates for the variation are the material density and/or the elastic moduli. The case of the varying material density, albeit for the isotropic case, was studied by Ran (1976). Here we deal with the variation of the elastic moduli. Semi-inverse vibration problem is formulated in which the mode shape is specified while the natural frequency is sought. The postulated mode shape is taken in the form of the static deflection of the isotropic circular plate studied long ago by Timoshenko and Woinowsky-Krieger (1959).

We show that heterogeneity of the plate allows for the following remarkable phenomena to take place: polar orthotropic plate possesses the mode shape that coincides with the static deflection of the homogeneous circular plate under uniformly distributed loading; closed form solution is obtained for the design of the heterogeneous plate with pre-selected fundamental of frequency.

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2. Derivation of governing differential equation

The relationship between the radial and circumferential bending moments with the transverse displacement w reads, in the polar coordinate system, in axisymmetric setting (Lekhnitskii, 1968):

$$M_r(r) = -D_r \left(w'' + \frac{v_\theta}{r} w' \right) \quad (1)$$

$$M_\theta(r) = -D_\theta \left(v_r w'' + \frac{1}{r} w' \right) \quad (2)$$

where M_r is the radial bending moment, M_θ the circumferential bending moment, D_r and D_θ are, respectively, radial and circumferential flexural rigidities, v_r = Poisson's ratio, prime denotes differentiation with respect to radial coordinate r . The equilibrium equation is

$$M_r + \frac{dM_r}{dr} r - M_\theta + Q_r r = 0 \quad (3)$$

Following Timoshenko and Woinowsky-Krieger (1959) the shearing force Q_r is expressed as

$$2\pi r Q_r = \int_0^r \int_0^{2\pi} q(\rho, \theta) \rho d\rho d\theta \quad (4)$$

where $q(r, \theta)$ is the distributed transverse load. We are considering the problem in axisymmetric setting, hence $q(r, \theta) = q(r)$. From Eq. (4) we get

$$Q_r = \int_0^r q(\rho) \rho d\rho \quad (5)$$

After substitution of Eqs. (1), (2) and (5) into (3) we arrive at

$$r D_r w''' - \left(-D_r - r \frac{dD_r}{dr} - v_\theta D_r + v_r D_\theta \right) w'' - \left(-v_\theta \frac{dD_r}{dr} + \frac{1}{r} D_\theta \right) w' = \int_0^r q(r) r dr \quad (6)$$

Differentiating the result respect to r , we obtain

$$\begin{aligned} r D_r w^{IV} + \left(2D_r + 2r \frac{dD_r}{dr} + v_\theta D_r - v_r D_\theta \right) w''' - \left[\frac{d}{dr} \left(-D_r - r \frac{dD_r}{dr} - v_\theta D_r + v_r D_\theta \right) - v_\theta \frac{dD_r}{dr} + \frac{1}{r} D_\theta \right] w'' \\ - \left[\frac{d}{dr} \left(-v_\theta \frac{dD_r}{dr} + \frac{1}{r} D_\theta \right) \right] w' = r q(r) \end{aligned} \quad (7)$$

To derive the equation of motion, one applies D'Alembert's principle and substitutes

$$q(r) = -\rho h \delta^2 \omega / \delta t^2 \quad (8)$$

into Eq. (7). In the resulting equation we let $w(r, t) = W(r) \sin \omega t$, where $W(r)$ is the mode shape, t the time, ρ the material density, h the thickness, and ω is the sought natural frequency. The final governing equation reads

$$\begin{aligned} r D_r W^{IV} + \left(2D_r + 2r \frac{dD_r}{dr} + v_\theta D_r - v_r D_\theta \right) W''' - \left[\frac{d}{dr} \left(-D_r - r \frac{dD_r}{dr} - v_\theta D_r + v_r D_\theta \right) - v_\theta \frac{dD_r}{dr} + \frac{1}{r} D_\theta \right] W'' \\ - \left[\frac{d}{dr} \left(-v_\theta \frac{dD_r}{dr} + \frac{1}{r} D_\theta \right) \right] W' = r \rho h \omega^2 W(r) \end{aligned} \quad (9)$$

For the isotropic plate $D_r = D_\theta = D$, $v_r = v_\theta = \nu$, and Eq. (9) reduces to Eq. (11.1) (Elishakoff, 2005).

3. Semi-inverse method of solution

We postulate the following mode shape

$$W(r) = (R^2 - r^2)^2 \quad (10)$$

that is proportional to the static displacement of the isotropic homogeneous circular plate of constant thickness under uniformly distributed load (Timoshenko and Woinowsky-Krieger, 1959).

In this section we consider the case of the constant material density $\rho = \text{const}$. The flexural rigidities are expressed as polynomials of fourth order

$$D_r(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4 \quad (11)$$

$$D_\theta(r) = k^2 D_r(r) \quad (12)$$

where k is taken as constant. Eq. (11) signifies that the circumferential flexural rigidity is proportional to the radial flexural rigidity. In subsequent analysis the Poisson's ratios v_r and v_θ are taken as constants.

The substitution of Eqs. (10)–(12) into Eq. (9) leads to the following polynomial equation:

$$A_0 + A_1 r + A_2 r^2 + A_3 r^3 + A_4 r^4 + A_5 r^5 = 0 \quad (13)$$

where A_j are following coefficients:

$$A_0 = -8R^2 b_1 - 8v_\theta R^2 b_1 + 4R^2 v_r k^2 b_1 + 4R^2 k^2 b_1 \quad (14)$$

$$A_1 = 72b_0 + 24v_\theta b_0 - 24v_r k^2 b_0 - 8k^2 b_0 - 24R^2 b_2 - 24R^2 v_\theta b_2 + 8v_r k^2 R^2 b_2 + 8k^2 R^2 b_2 - \rho h \omega^2 R^4 \quad (15)$$

$$A_2 = 144b_1 + 48v_\theta b_0 - 36v_r k^2 b_1 - 12k^2 b_1 - 48R^2 b_3 - 48R^2 v_\theta b_3 + 12v_r k^2 R^2 b_3 + 12k^2 R^2 b_3 \quad (16)$$

$$A_3 = 240b_2 + 80v_\theta b_2 - 48v_r k^2 b_2 - 16k^2 b_2 - 80R^2 b_4 - 80R^2 v_\theta b_4 - 80v_r k^2 R^2 b_4 + 16k^2 R^2 b_4 + 2\rho h \omega^2 R^2 \quad (17)$$

$$A_4 = 360b_3 + 120v_\theta b_3 - 60v_r k^2 b_3 - 20k^2 b_3 \quad (18)$$

$$A_5 = 504b_4 + 168v_\theta b_4 - 72v_r k^2 b_4 - 24k^2 b_4 - \rho h \omega^2 \quad (19)$$

From Eq. (19) we get the relationship between the natural frequency squared ω^2 and the coefficient b_4 :

$$\omega^2 = 24(21 + 7v_\theta - k^2 - 3k^2 v_r) b_4 / \rho h \quad (20)$$

Eq. (18) shows that b_3 vanishes identically. The equation resulting from substitution of Eq. (20) into Eq. (17), results in the formula for b_2 , as related to b_4

$$b_2 = -2R^2 \frac{(k^2 - 8v_\theta - 29 + 4k^2 v_r)}{3k^2 v_r + k^2 - 5v_\theta - 15} b_4 \quad (21)$$

Eq. (14) leads to the conclusion that $b_1 = 0$. The equation resulting for substitution of Eqs. (20) and (21) into Eq. (15) yields to the expression for b_0 :

$$b_0 = \frac{Q_1 R^4}{(k^2 + 3k^2 v_r - 5v_\theta - 15)(k^2 - 9 - 3v_\theta + 3k^2 v_r)} b_4 \quad (22)$$

where

$$Q_1 = -242k^2 v_r - 68k^2 v_r v_\theta + 408v_\theta - 44k^2 b + 8k^4 v_r - 14k^2 v_\theta + 57v_\theta^2 + 19k^4 v_r^2 + k^4 + 771 \quad (23)$$

Thus, we arrive at the following expression for the flexural rigidity:

$$D_r(r) = \left[\frac{Q_1 R^4}{(k^2 + 3k^2 v_r - 5v_\theta - 15)(k^2 - 9 - 3v_\theta + 3k^2 v_r)} - \frac{2R^2(k^2 - 8v_\theta - 29 + 4k^2 v_r)}{3k^2 v_r + k^2 - 5v_\theta - 15} r^2 + r^4 \right] b_4 \quad (24)$$

or, in the view of the relationship (Lekhnitskii, 1968)

$$v_\theta = k^2 v_r \quad (25)$$

we get

$$D_r(r) = \frac{1}{(2k^2 v_r - k^2 + 15)(k^2 - 9)} \left[-8R^4 k^4 v_r^2 - 166R^4 k^4 v_r + 44R^4 k^2 + 6R^4 k^4 v_r - 771R^4 - R^4 k^4 \right. \\ \left. + (-76k^2 + 2R^2 k^4 + 522R^2 + 72R^2 k^2 v_r - 8R^2 k^4 v_r) r^2 + (-18v_r + 2k^4 v_r + 24k^2 - k^4 - 135) r^4 \right] b_4 \quad (26)$$

Thus, the heterogeneous plate of the flexural rigidity specified by Eq. (26) possesses the fundamental natural frequency given in Eq. (20). Eqs. (20) and (26) show that as a by-product of the semi-inverse analysis we get the possibility of tailoring, i.e. designing a plate that has a pre-selected natural frequency Ω .

Indeed, Eq. (20) can be put in the form

$$\omega^2 = A_1 b_4 \quad (27)$$

where A_1 depends on the problem parameters. For mode shape in Eq. (10), the value A_1 equals

$$A_1 = 24(21 + 7v_\theta - k^2 - 3k^2 v_r) / \rho h \quad (28)$$

as obtained from Eq. (20). From our demand $\omega = \Omega$, we obtain the value of coefficient by facilitate the vibration tailoring

$$b_4 = \Omega^2 / A_1 \quad (29)$$

Once this value is substituted into the flexural rigidity expression, namely, in Eq. (26), the unique plate is obtained, that possesses the predetermined natural frequency Ω . Specifically, the flexural rigidity of the plate with fundamental natural frequency Ω reads

$$D_r(r) = \frac{1}{(2k^2 v_r - k^2 + 15)(k^2 - 9)} \left[-8R^4 k^4 v_r^2 - 166R^4 k^4 v_r + 44R^4 k^2 + 6R^4 k^4 v_r - 771R^4 - R^4 k^4 \right. \\ \left. + (-76k^2 + 2R^2 k^4 + 522R^2 + 72R^2 k^2 v_r - 8R^2 k^4 v_r) r^2 + (-18v_r + 2k^4 v_r + 24k^2 - k^4 - 135) r^4 \right] \frac{\Omega^2}{A_1} \quad (30)$$

Thus, analytical expression incorporates in itself the value of the desired natural frequency Ω .

The Fig. 1 shows the variation of the radial flexural rigidity Coefficient:

$$d = D_r(r)/b_4 R^4 \quad (31)$$

with the non-dimensional radial coordinate for different values of k . Larger values of $d(r)$ are associated with greater anisotropy measure, namely k .

4. Parabolic mode shape

In this case the mode shape represents a second-order polynomial

$$W(r) = (R - r)^2 \quad (32)$$

The result of substitution of Eq. (32) in conjunction of Eqs. (11) and (12) yields the equation

$$\sum_{j=0}^5 B_j r^j = 0 \quad (33)$$

where

$$B_0 = -2Rk^2 b_0 \quad (34)$$

$$B_1 = 0 \quad (35)$$

$$B_2 = 2k^2 b_2 R + 4v_0 b_1 - 4Rv_0 b_2 - 2k^2 b_1 + 4b_1 - 2v_r k^2 b_1 \quad (36)$$

$$B_3 = -4k^2 b_2 - 12v_0 b_3 R + 4k^2 b_3 R + 12b_2 + 12v_0 b_2 - 4v_r k^2 R b_2 - \rho h \omega^2 R^2 \quad (37)$$

$$B_4 = -24b_4 v_0 R - 6k^2 b_3 + 24b_3 - 6k^2 b_3 v_r + 24b_3 v_0 + 6k^2 R b_4 + 2\rho h \omega^2 R \quad (38)$$

$$B_5 = 40b_4 v_0 - 8k^2 b_4 - 40b_4 - 8k^2 b_4 v_r - \rho h \omega^2 \quad (39)$$

From Eq. (39) we get the expression for the natural frequency squared:

$$\omega^2 = 8(5 + 5v_0 - k^2 - k^2 v_r) b_4 / \rho h \quad (40)$$

Substitution of Eq. (40) into Eq. (38) leads to

$$b_3 = -\frac{1}{3} \frac{(5k^2 + 8k^2 v_r - 28v_0 - 40)}{(-4v_0 + k^2 + k^2 v_r - 4)} R b_4 \quad (41)$$

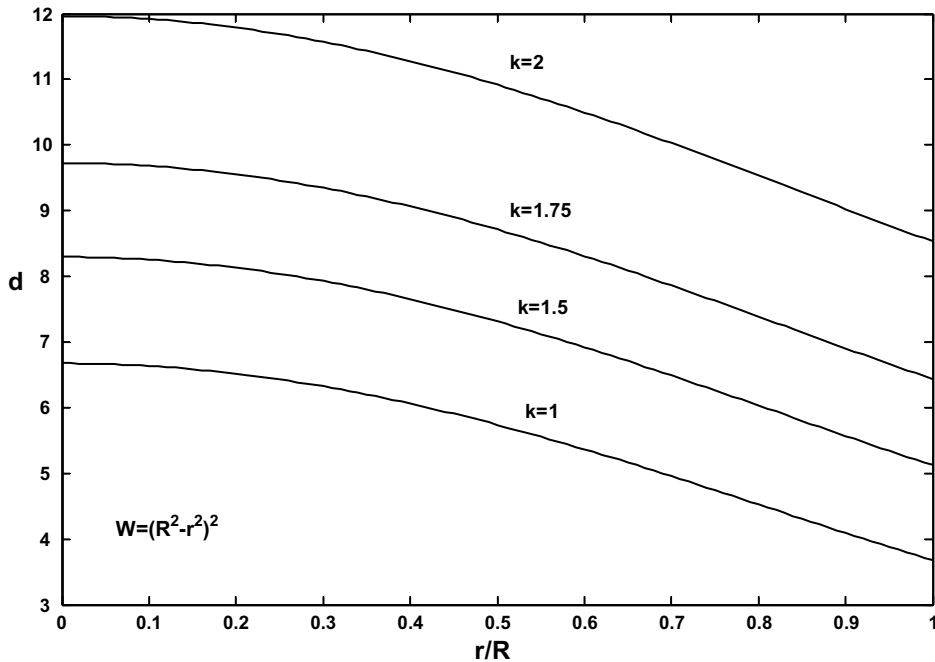


Fig. 1. Variation of d vs. non-dimensional radial coordinate r/R for various values of k , and $v_r = 0.35$.

Substituting Eqs. (40) and (41) into Eq. (47) we get

$$b_2 = \frac{1}{3(k^2 - 4v_\theta + k^2v_r - 4)(k^2 - 3 - 3v_\theta + k^2v_r)} (-30k^2v_rv_\theta + 4k^4v_r + 6k^4v_r^2 - 54k^2v_r + 36v_\theta^2 - 11v_\theta k^2 + 120v_\theta + k^4 - 14k^2 + 120)R^2b_4 \quad (42)$$

Eqs. (36) and (42) yield

$$b_1 = \frac{Q_2}{3(k^2 - 4v_\theta + k^2v_r - 4)(k^2 - 3 - 3v_\theta + k^2v_r)(k^2 + k^2v_r - 2v_\theta - 2)} R^3b_4 \quad (43)$$

where

$$Q_2 = -30k^2v_rv_\theta + 4k^4v_r + 6k^4v_r^2 - 54k^2v_r + 36v_\theta^2 - 11v_\theta k^2 + 120v_\theta + k^4 - 14k^2 + 120 \quad (44)$$

From Eq. (34) we derive

$$b_0 = 0 \quad (45)$$

The flexural rigidity $D_r(r)$ is obtained by substituting Eqs. (41)–(45) into Eq. (11)

$$D_r(r) = \left[\frac{1}{3(k^2 - 4v_\theta + k^2v_r - 4)(k^2 - 3v_\theta + k^2v_r - 3)(k^2 - 2v_\theta + k^2v_r - 2)} (-2v_\theta + k^2) \right. \\ \times (36v_\theta^2 - 11v_\theta k^2 - 30v_\theta v_r k^2 + 120v_\theta + 4k^4v_r^2 + 6k^4v_r - 54k^2v_r + k^4 - 14k^2 + 120)R^3r \\ + \frac{(36v_\theta^2 - 11v_\theta k^2 - 30v_\theta v_r k^2 + 120v_\theta + 4k^4v_r^2 + 6k^4v_r - 54k^2v_r + k^4 - 14k^2 + 120)}{3(k^2 - 4v_\theta + k^2v_r - 4)(k^2 - 3v_\theta + k^2v_r - 3)} R^2r^2 \\ \left. - \frac{(5k^2 - 28v_\theta + 8k^2v_r - 40)}{3(k^2 - 4v_\theta + k^2v_r - 4)} Rr^3 + r^4 \right] b_4 \quad (46)$$

Due to the fact that $b_0 = 0$, $D_r(0)$ and $D_\theta(0)$ vanish. For $v_r = 0.35$ the minimum value of k resulting in non-negative D_r over $r/R \in [0;1]$ equals 1.75412, as is shown in the Fig. 2. Fig. 3 depicts the variation of the radial flexural rigidity coefficient $d_r = D_r(r)/b_4R^4$ with the non-dimensional radial coordinate r/R for $k = 2; 2.5; 2.75; 3$ and $v_r = 0.35$. Note that when $k = 1$, for a Poisson's ratio that differs from $1/2$, the plate does not acquire the parabolic shape in Eq. (32), as was demonstrated in (Elishakoff, 2005). Like in Section 3 there is possibility for the vibration tailoring – construction the plate with preselected fun-

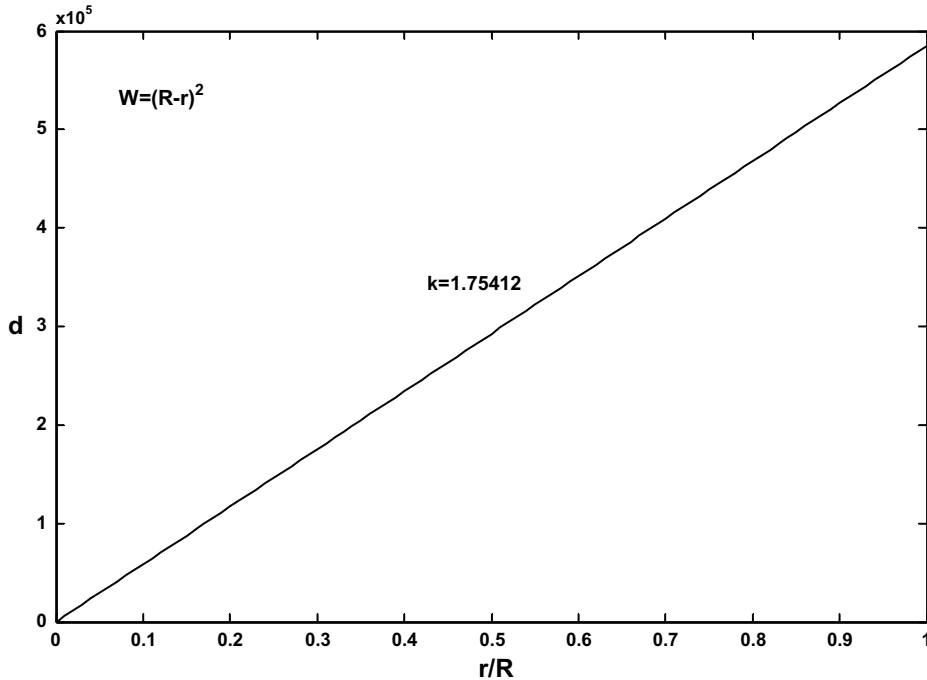


Fig. 2. Variation of d vs. non-dimensional radial coordinate r/R for $k = 1.75412$, and $v_r = 0.35$.

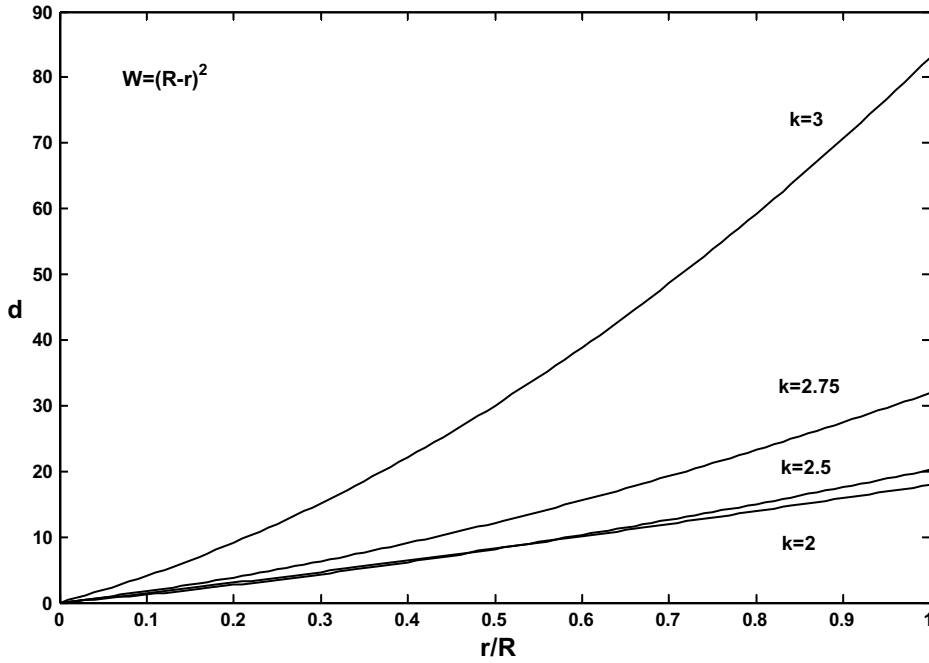


Fig. 3. Variation of d vs. non-dimensional radial coordinate r/R for various values of k , and $\nu_r = 0.35$.

damental frequency Ω . The analytical expression for flexural rigidity $D_r(r)$ of the plate with fundamental natural frequency Ω reads:

$$D_r(r) = \left[\frac{1}{3(k^2 - 4\nu_\theta + k^2\nu_r - 4)(k^2 - 3\nu_\theta + k^2\nu_r - 3)(k^2 - 2\nu_\theta + k^2\nu_r - 2)} (-2\nu_\theta + k^2) \right. \\ \times (36\nu_\theta^2 - 11\nu_\theta k^2 - 30\nu_\theta\nu_r k^2 + 120\nu_\theta + 4k^4\nu_r^2 + 6k^4\nu_r - 54k^2\nu_r + k^4 - 14k^2 + 120)R^3r \\ \left. + \frac{(36\nu_\theta^2 - 11\nu_\theta k^2 - 30\nu_\theta\nu_r k^2 + 120\theta_0 4k^4\nu_r^2 + 6k^4\nu_r - 54k^2\nu_r + k^4 - 14k^2 + 120)R^2r^2}{3(k^2 - 4\nu_\theta + k^2\nu_r - 4)(k^2 - 3\nu_\theta + k^2\nu_r - 3)} \right. \\ \left. - \frac{(5k^2 - 28\nu_\theta + 8k^2\nu_r - 40)Rr^3 + r^4}{3(k^2 - 4\nu_\theta + k^2\nu_r - 4)} \right] \frac{\Omega^2}{A_2} \quad (47)$$

where A_2 is obtained from Eq. (40)

$$A_2 = 8(5 + 5\nu_\theta - k^2 - k^2\nu_r)/\rho h \quad (48)$$

5. Two cubic mode shapes

Now we check if the following expression of cubic mode shape

$$W(r) = (R - r)^3 \quad (49)$$

can be possessed by the polar orthotropic plate. We are following the same procedure as in the previous cases. The substitution of Eqs. (49), (11) and (12) into Eq. (9) leads to the equation

$$\sum_{j=0}^6 C_j r^j = 0 \quad (50)$$

where

$$C_0 = -3R^2 k^2 b_0 \quad (51)$$

$$C_1 = 0 \quad (52)$$

$$C_2 = -6k^2 b_1 R + 6k^2 \nu_r b_0 + 3k^2 R^2 b_2 + 3k^2 b_0 - 6k^2 b_1 R \nu_r + 12R b_1 - 6b_2 \nu_\theta R^2 + 12b_1 \nu_\theta R - 6\nu_\theta b_0 - 12b_0 \quad (53)$$

$$C_3 = -12k^2 R b_2 + 36b_2 R + 36\nu_\theta b_2 R - 18b_3 \nu_\theta R^2 + 6k^2 b_3 R^2 - 36b_1 - 18b_1 \nu_\theta + 12k^2 \nu_r b_1 - 12k^2 b_2 R \nu_r + 6k^2 b_1 - \rho h \omega^2 R^3 \quad (54)$$

$$C_4 = 72b_3 \nu_\theta R - 36\nu_\theta R^2 b_4 + 72b_3 R - 36b_2 \nu_\theta - 72b_2 + 9k^2 b_2 - 18k^2 b_3 R - 18k^2 b_3 R \nu_r + 18k^2 b_2 \nu_r + 9k^2 R^2 b_4 + 3\rho h \omega^2 R^2 \quad (55)$$

$$C_5 = 120b_4R + 24k^2b_3v_r - 24k^2b_4Rv_r - 60b_3v_\theta + 12k^2b_3 + 120b_4v_\theta R - 120b_3 - 24k^2b_4R - 3\rho h\omega^2R \quad (56)$$

$$C_6 = 30k^2b_4v_r - 180b_4 - 90b_4v_\theta + 15k^2b_4 + \rho h\omega^2 \quad (57)$$

From Eq. (57) we get the expression of the natural frequency squared ω^2 with respect of the coefficient b_4 :

$$\omega^2 = 15(12 + 6v_\theta - k^2 - 2k^2v_r)b_4/\rho h \quad (58)$$

The coefficients of flexural rigidity are as follows:

$$b_0 = -\frac{Q_3}{(k^2 - 5v_\theta + 2k^2v_r - 10)(k^2 - 8 - 4v_\theta + 2k^2v_r)(k^2 + 2k^2v_r - 3v_\theta - 6)(k^2 - 2v_\theta - 4 + 2k^2v_r)}R^4b_4 \quad (59)$$

$$b_1 = -\frac{Q_4}{4(k^2 - 5v_\theta + 2k^2v_r - 10)(k^2 - 8 - 4v_\theta + 2k^2v_r)(k^2 + 2k^2v_r - 3v_\theta - 6)}R^3b_4 \quad (60)$$

$$b_2 = \frac{1}{2(k^2 - 5v_\theta + 2k^2v_r - 10)(k^2 - 8 - 4v_\theta + 2k^2v_r)}(-66k^2v_rv_\theta + 7k^4v_r + 18k^4v_r^2 - 212k^2v_r + 60v_\theta^2 - 14v_\theta k^2 + 360v_\theta + k^4 - 32k^2 + 640)R^2b_4 \quad (61)$$

$$b_3 = -\frac{1}{4}\frac{(7k^2 + 22k^2v_r - 50v_\theta - 140)}{(k^2 - 5v_\theta + 2k^2v_r - 20)}Rb_4 \quad (62)$$

where

$$Q_3 = -2664k^2v_rv_\theta + 816k^4v_r + 544k^4v_r^2 - 1776k^2v_r + 2400v_\theta^2 - 708v_\theta k^2 + 4800v_\theta - 172k^4 - 1776k^2 + 1920 + 600v_\theta^3 - k^6 + 25k^6v_r - 152k^6v_r^2 - 76k^6v_r^3 - 35v_\theta k^4 - 218v_\theta^2k^2 + 544k^4v_r^2v_\theta + 376k^4v_rv_\theta - 1052k^2v_rv_\theta^2 + 116k^4v_r^2v_\theta^2 + 65k^4v_rv_\theta^3 - 142k^2v_rv_\theta^3 - 45v_\theta k^6v_r^2 - 38v_\theta k^6v_r^3 - 30v_\theta^3k^2 + 60v_\theta^4 + 10k^8v_r^3 + 4k^8v_r^4 \quad (63)$$

$$Q_4 = -224k^2v_rv_\theta + 316k^4v_r - 136k^4v_r^2 - 54k^2v_r + 720v_\theta^2 - 580v_\theta k^2 + 960v_\theta + 36k^4 - 1104k^2 - 1920 + 120v_\theta^3 - k^6 - 8k^6v_r - 24k^6v_r^2 + 8k^6v_r^3 + 17v_\theta k^4 - 102v_\theta^2k^2 + 12k^4v_r^2v_\theta + 100k^4v_rv_\theta - 116k^2v_rv_\theta^2 \quad (64)$$

with $v_\theta = k^2v_r$ the final expression of the flexural rigidity becomes, in view of Eq. (58)

$$D_r(r) = \left[-\frac{(280k^4v_r^2 + 3024k^2v_r - 1776k^2 - 172k^4 + 108k^4v_r + 16k^6v_r^3 - 10k^6v_r + 6k^6v_r^2)R^4}{(k^2 - 10 - 3k^2v_r)(k^2 - 2k^2v_r - 8)(k^2 - 6 - k^2v_r)(k^2 - 4)} - \frac{(12k^4v_r^2 - 7k^4v_r + 66k^2v_r + k^4 - 14k^2 + 120)(-2k^2v_r - 6)}{4(k^2 - 3k^2v_r - 10)(k^2 - 8 - 2k^2v_r)(k^2 - k^2v_r - 6)}R^3r + \frac{(12k^4v_r^2 - 7k^4v_r + 148k^2v_r + k^4 - 32k^2 + 640)}{2(k^2 - 3k^2v_r - 10)(k^2 - 8 - 2k^2v_r)}R^2r^2 - \frac{(7k^2 - 28k^2v_r - 140)}{4(k^2 - 3k^2v_r - 10)}Rr^3 + r^4 \right] \frac{\Omega^2}{A_3}, \quad (65)$$

$$A_3 = 15(12 + 6v_\theta - k^2 - 2k^2v_r)/\rho h$$

It should be noted that not for all values of k the physically realizable solutions exist; namely, for $v_r = 0.35$ the minimum value of k resulting in positive D_r over $r/R \in [0;1]$ equals 3.039, as shown in the Fig. 4. In Fig. 5, the flexural rigidity coefficients are depicted for values $k = 3.5; 4.2; 4.4$ and 4.6 . It is remarkable to note that at $r/R = 0.27$ the values of $d(r)$ corresponding to $k = 3.5$ and 4.6 intersect. Likewise, at $r/R = 0.92$ the $d(r)$ values associated with $k = 3.5$ and $k = 4.4$ intersect. This shows, that there is no monotonic dependence of the coefficient $d(r)$ with the rigidity ratio k .

Consider now an alternative cubic mode shape

$$W(r) = R^3 - 3r^2R + 2r^3 \quad (66)$$

The substitution of Eqs. (66), (11) and (12) into Eq. (9) leads to the equation

$$\sum_{j=0}^4 D_j r^j = 0 \quad (67)$$

where

$$D_0 = 6k^2b_1R - 12k^2b_0v_r + 24b_0 + 12b_0v_\theta - 6k^2b_0 + 6k^2b_1v_rR - 12b_1v_\theta R - 12b_1R \quad (68)$$

$$D_1 = 12k^2b_2v_rR - 36b_2R + 36b_1v_\theta - 12k^2b_1 + 12k^2b_2R + 72b_1 - 36b_2v_\theta R - 24k^2b_1v_r - \rho h\omega^2R^3 \quad (69)$$

$$D_2 = 18k^2b_3v_rR - 18k^2b_2 + 72v_\theta b_2 - 72Rb_3 + 18k^2Rb_3 - 36k^2b_2v_r - 72b_3v_\theta^R + 144b_2 \quad (70)$$

$$D_3 = 24k^2v_rR + 240b_3 - 24k^2b_3 - 48k^2b_3v_r - 120b_4R + 120b_3v_\theta - 120b_4Rv_\theta + 24k^2b_4R + \rho h\omega^2R \quad (71)$$

$$D_4 = -30k^2b_4 - 60k^2b_4v_r + 180b_4v_\theta + 360b_4 - 2\rho h\omega^2 \quad (72)$$

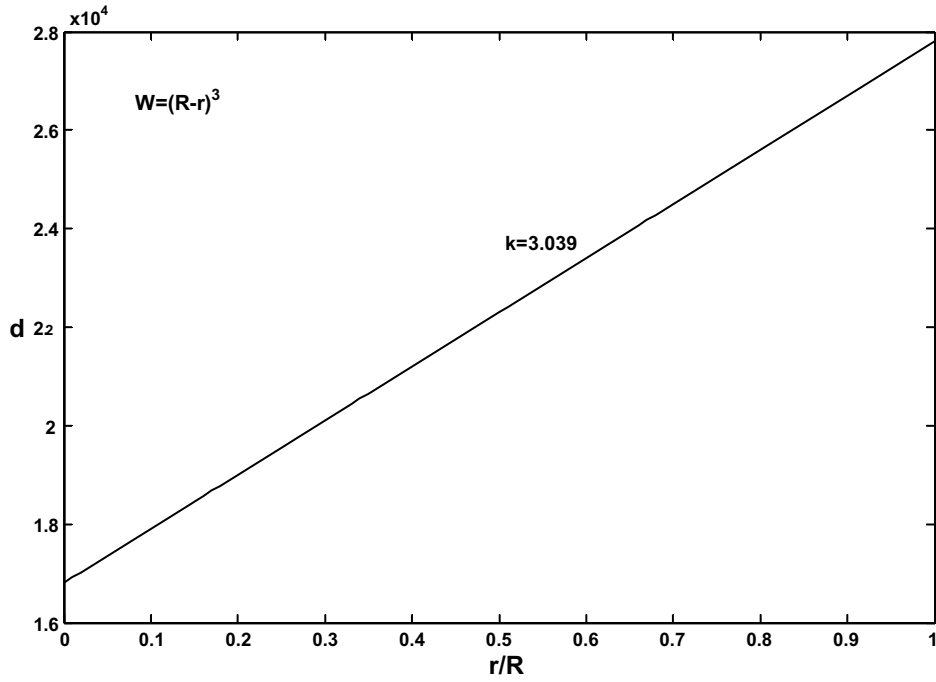


Fig. 4. Variation of d vs. non-dimensional radial coordinate r/R for $k = 3.039$, and $\nu_r = 0.35$.

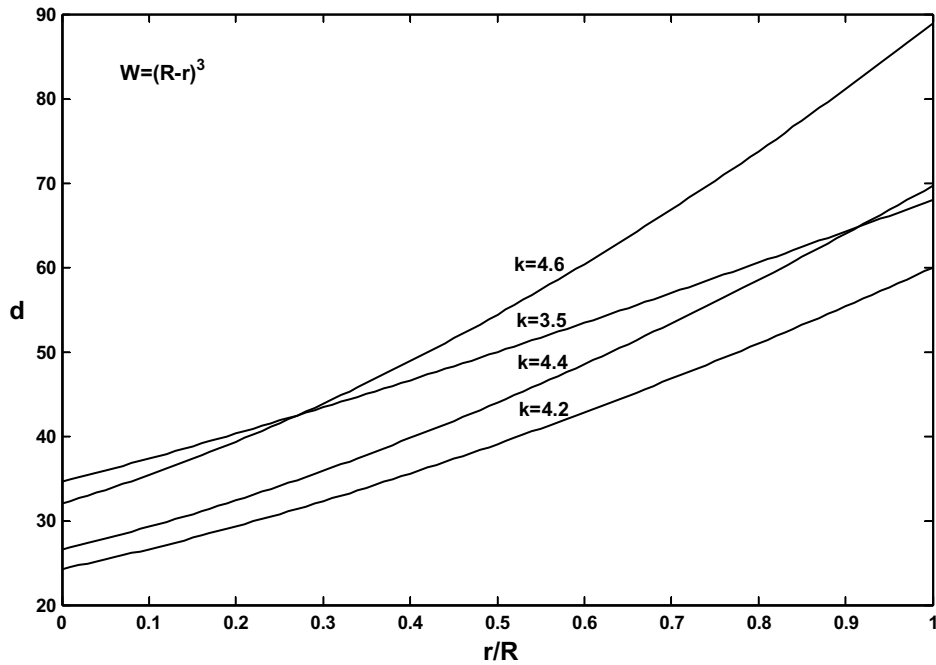


Fig. 5. Variation of d vs. non-dimensional radial coordinate r/R for various values of k , and $\nu_r = 0.35$.

From Eq. (72) we get the expression of the natural frequency squared ω^2 :

$$\omega^2 = 15(12 + 6\nu_\theta - k^2 - 2k^2\nu_r)b_4/\rho h \quad (73)$$

The coefficients of flexural rigidity are as follows

$$b_0 = \frac{Q_5}{8(k^2 - 5v_\theta + 2k^2v_r - 10)(k^2 - 8 - 4v_\theta + 2k^2v_r)(k^2 + 2k^2v_r - 3v_\theta - 6)(k^2 - 2v_\theta - 4 + 2k^2v_r)} R^4 b_4 \quad (74)$$

$$b_1 = \frac{Q_6}{8(k^2 - 5v_\theta + 2k^2v_r - 10)(k^2 - 8 - 4v_\theta + 2k^2v_r)(k^2 + 2k^2v_r - 3v_\theta - 6)} R^3 b_4 \quad (75)$$

$$b_2 = -\frac{(7k^2 - 50v_\theta + 22k^2v_r - 140)(k^2 - 4 + k^2v_r - 4v_\theta)}{8(k^2 - 5v_\theta + 2k^2v_r - 10)(k^2 - 8 - 4v_\theta + 2k^2v_r)} R^2 b_4 \quad (76)$$

$$b_3 = -\frac{1}{8} \frac{(7k^2 + 22k^2v_r - 50v_\theta - 140)}{(k^2 - 5v_\theta + 2k^2v_r - 10)} R b_4 \quad (77)$$

where

$$Q_5 = (4676k^2v_r + 1896k^2 - 10440v_\theta + 1896k^2 - 297k^4v_\theta v_r - 396v_r^2v_\theta k^4 + 1462v_\theta k^2 - 4320v_\theta^2 - 717k^4v_r - 906k^4v_r^2 - 51v_\theta k^4 + 306k^2v_\theta^2 + 24k^6v_r + 69k^6v_r^2 - 111k^4 + 3k^6 - 600v_\theta^3 + 866k^2v_rv_\theta^2 + 58k^6v_r^3 - 7920)(-2v_\theta - 2 + k^2v_r + k^2) \quad (78)$$

$$Q_6 = 4062k^2v_rv_\theta - 717k^4v_r - 906k^4v_r^2 + 4676k^2v_r - 4320v_\theta^2 + 1462v_\theta k^2 - 10440v_\theta - 111k^4 + 1896k^2 - 7920 - 600v_\theta^3 + 3k^6 + 24k^6v_r + 69k^6v_r^2 + 58k^6v_r^3 - 51v_\theta k^4 + 306v_\theta^2k^2 - 396k^4v_r^2v_\theta - 297k^4v_rv_\theta + 866k^2v_rv_\theta^2 \quad (79)$$

with $v_\theta = k^2v_r$ the final expression of the flexural rigidity is, in the view of Eq. (73)

$$D_r(r) = \left[\frac{(-1164k^4v_r^2 - 5764k^2v_r + 1896k^2 - 111k^4 + 745k^4v_r - 72k^6v_r^3 - 27k^6v_r + 78k^6v_r^2 + 3k^6)(k^2 - k^2v_r - 2)}{8(k^2 - 10 - 3k^2v_r)(k^2 - 2k^2v_r - 8)(k^2 - 6 - k^2v_r)(k^2 - 4)} R^4 \right. \\ \left. + \frac{(-1164k^4v_r^2 + 745k^4v_r - 5764k^2v_r - 111k^4 + 1896k^2 - 27k^6v_r + 78k^6v_r^2 - 72k^6v_r^3 - 7920)}{8(k^2 - 3k^2v_r - 10)(k^2 - 8 - 2k^2v_r)(k^2 - k^2v_r - 6)} R^3 r \right. \\ \left. - \frac{(7k^2 - 28k^2v_r - 140)(k^2 - 3k^2v_r - 4r)}{8(k^2 - 3k^2v_r - 10)(k^2 - 8 - 2k^2v_r)} R^2 r^2 - \frac{(7k^2 - 28k^2v_r - 140)}{8(k^2 - 3k^2v_r - 10)} R r^3 + r^4 \right] \frac{\Omega^2}{A_4}, \\ A_4 = 15(12 + 6v_\theta - k^2 - 2k^2v_r)/\rho h \quad (80)$$

Fig. 6 depicts the flexural rigidity coefficient d for three different values of k . Note that for k taking values 2 , $\sqrt{6/(1 - v_r)}$, $\sqrt{8/(1 - 2v_r)}$, or $\sqrt{10/(1 - 3v_r)}$ the denominator in Eq. (74) becomes zero, and the solution must be modified for these particular cases. The case $k = 3$, $v_r = 0.35$ is shown in Fig. 7.

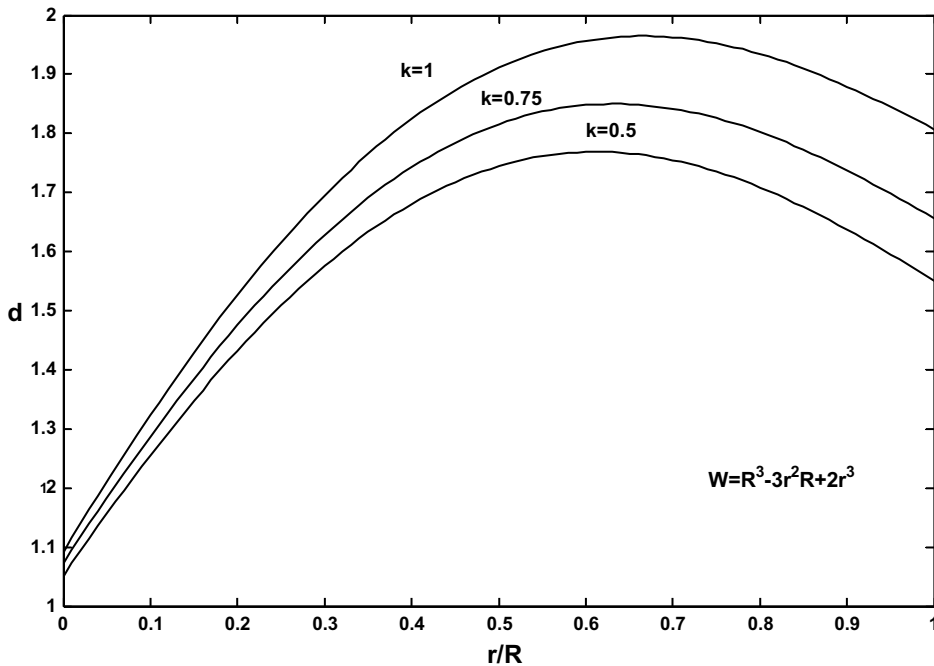


Fig. 6. Variation of d vs. non-dimensional radial coordinate r/R for various values of k , and $v_r = 0.35$.

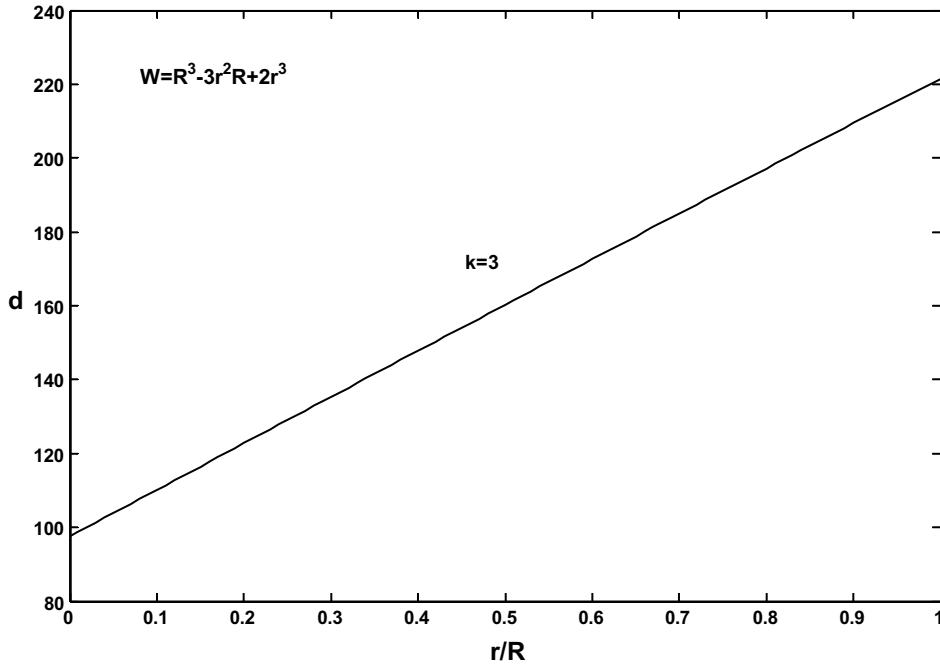


Fig. 7. Variation of d vs. non-dimensional radial coordinate r/R for $k=3$, and $\nu_r=0.35$.

6. Two alternative quartic mode shapes

Note that the expression for the static displacement of the uniform homogeneous circular plate contains in it a quartic polynomial, given in Eq. (10). Natural question arises if other quartics may serve as exact mode shape. We investigate the following expression:

$$W(r) = (R - r)^4 \quad (81)$$

as a candidate mode shape.

The substitution of Eqs. (81), (11) and (12) into Eq. (9) leads to the equation

$$\sum_{j=0}^7 E_j r^j = 0 \quad (82)$$

where

$$E_0 = -4R^3 k^2 b_0 \quad (83)$$

$$E_1 = 0 \quad (84)$$

$$E_2 = 4k^2 b_2 R^3 + 12k^2 b_0 R + 24k^2 \nu_r R b_0 - 48R b_0 - 8R^3 b_2 \nu_\theta + 24b_1 R^2 \nu_\theta - 24\nu_\theta R b_0 + 24k^2 b_0 R \nu_r - 48b_0 R - 8b_2 R^2 \nu_\theta + 24b_1 R^2 \nu_\theta - 24b_0 \nu_\theta R - 12k^2 b_1 R^2 \nu_r - 12k^2 b_1 R^2 + 24b_1 R^2 \quad (85)$$

$$E_3 = 24\nu_\theta b_0 + 48\nu_r k^2 R b_1 - 24k^2 b_2 R^2 \nu_r + 8k^2 b_3 R^3 - 24k^2 b_2 R^2 + 72b_2 R^2 - 72b_1 \nu_\theta R - 144b_1 R - 72b_0 + 24k^2 b_1 R - 8k^2 b_0 + 72b_2 R^2 \nu_\theta - 24k^2 b_0 \nu_r - 24b_3 R^3 \nu_\theta - \rho h \omega^2 R^4 \quad (86)$$

$$E_4 = 48k^2 b_3 R + 80b_2 \nu_\theta - 16k^2 b_2 + 240b_2 + 96k^2 b_3 R \nu_r + 240b_4 R^2 - 48k^2 b_4 R^2 - 240R \nu_\theta b_3 - 48k^2 b_2 \nu_r - 48k^2 b_4 R^2 \nu_r + 240b_4 R^2 \nu_\theta - 480b_3 R - 6\rho h \omega^2 R^2 \quad (87)$$

$$E_5 = 360b_3 + 120b_3 \nu_\theta - 60k^2 b_3 \nu_r - 360k^2 R \nu_r b_4 + 120k^2 R b_4 \nu_r + -710b_4 R - 20k^2 b_3 + 60k^2 b_4 R + 4\rho h R \omega^2 \quad (88)$$

$$E_6 = 168b_4 \nu_\theta - 72k^2 \nu_r b_4 + 504b_4 - 24k^2 b_4 - \rho h \omega^2 \quad (89)$$

From Eq. (89) we get the expression of the natural frequency squared ω^2 :

$$\omega^2 = 24(21 + 7\nu_\theta - k^2 - 3k^2 \nu_r) b_4 / \rho h \quad (90)$$

The coefficients of flexural rigidity are as follows:

$$b_0 = -\frac{9Q_7}{5(k^2 - 6v_0 + 3k^2v_r - 18)(k^2 - 15 - 5v_0 + 3k^2v_r)(k^2 + 3k^2v_r - 4v_0 - 12)(k^2 - 3v_0 - 9 + 3k^2v_r)}R^4b_4 \quad (91)$$

$$b_0 = -\frac{Q_8}{5(k^2 - 6v_\theta + 3k^2v_r - 18)(k^2 - 15 - 5v_\theta + 3k^2v_r)(k^2 + 3k^2v_r - 4v_\theta - 12)}R^4b_4 \quad (92)$$

$$b_2 = \frac{3}{5(-6v_\theta + k^2 + 3k^2v_r - 18)(-15 - 5v_\theta + 3k^2v_r + k^2) + 90v_\theta^2 - 17v_\theta k^2 + 780v_\theta + k^4 - 56k^2 + 1980)R^2b_4}(-144k^2v_rv_\theta + 10k^4v_r + 36k^4v_r^2 - 522k^2v_r) \quad (93)$$

$$b_3 = -\frac{3}{5} \frac{(3k^2 + 14k^2 v_r - 26v_\theta - 108)}{(k^2 - 6v_\theta + 3k^2 v_r - 18)} Rb_4 \quad (94)$$

where

$$Q_7 = -8586k^2 v_r v_\theta + 2652k^4 v_r + 1989k^4 v_r^2 - 8586k^2 v_r + 7200v_\theta^2 - 2448v_\theta k^2 + 21600v_\theta$$

$$- 399k^4 - 8586k^2 + 9720 + 1200v_\theta^3 - k^6 + 42k^6 v_r - 360k^6 v_r^2 - 216k^6 v_r^3 - 54v_\theta k^4 - 472v_\theta^2 k^2 + 1326k^4 v_r^2 v_\theta$$

$$+ 838k^4 v_r v_\theta - 2292k^2 v_r v_\theta^2 + 191k^4 v_r^2 v_\theta^2 + 94k^4 v_r v_\theta^2 - 208k^2 v_r v_\theta^3 - 72v_\theta k^6 v_r^2 - 72v_\theta k^6 v_r^3 - 40v_\theta^3 k^2 + 80v_\theta^4 + 18k^8 v_r^3 + 9k^8 v_r^4 \quad (95)$$

$$Q_8 = 1098k^2 v_r v_\theta + 972k^4 v_r - 1134k^4 v_r^2 + 8532k^2 v_r + 720v_\theta^2 - 1494v_\theta k^2 - 2160v_\theta + 66k^4 \\ - 4770k^2 - 25920 + 120v_\theta^3 - k^6 - 12k^6 v_r - 54k^6 v_r^2 + 54k^6 v_r^3 + 21v_\theta k^4 - 158v_\theta^2 k^2 - 108k^4 v_r^2 v_\theta + 186k^4 v_r v_\theta - 42k^2 v_r v_\theta^2 \quad (96)$$

with $v_\theta = k^2 v_r$ the final expression of the flexural rigidity becomes, with Eq. (90) taken into account

$$D_r(r) = \left[-\frac{9(-399k^4 - 8586k^2 + 204k^4v_r + 603k^4v_r^2 + 13014k^2v_r + 9720 + 6k^6v_r^2 - 12k^6v_r + 18k^6v_r^3)}{5(k^2 - 3k^2v_r - 18)(k^2 - 15 - 2k^2v_r)(k^2 - 12 - k^2v_r)(k^2 - 9)}R^4 \right. \\ - \frac{1(66k^4 - 4770k^2 - 522k^4v_r + 684k^4v_r^2 + 6372k^2v_r - 26k^6v_r^2 + 9k^6v_r + 24k^6v_r^3 - k^6 - 25920)}{5(k^2 - 3k^2v_r - 18)(k^2 - 15 - 2k^2v_r)(k^2 - 12 - k^2v_r)}R^3r \\ \left. + \frac{3}{5} \frac{(-56k^2 + 12k^4v_r^2 - 7k^4v_r + k^4 + 258k^2v_r + 1980)}{(k^2 - 3k^2v_r - 18)(k^2 - 15 - 2k^2v_r)}R^2r^2 - \frac{3}{5} \frac{(3k^2 - 12k^2v_r - 108)}{(k^2 - 3k^2v_r - 18)}Rr^3 + r^4 \right] \frac{\Omega^2}{A_5}, \\ A_5 = 24(21 + 7v_0 - k^2 - 3k^2v_r)/\rho h \quad (97)$$

Fig. 8 depicts the variation of d versus r/R . Note that for the isotropic case, $k = 1$ and $\nu_r = 0.35$ the function $d(r)$ takes negative values; this implies that for the isotropic case the Eq. (74) cannot serve the mode shape of the plate. For the

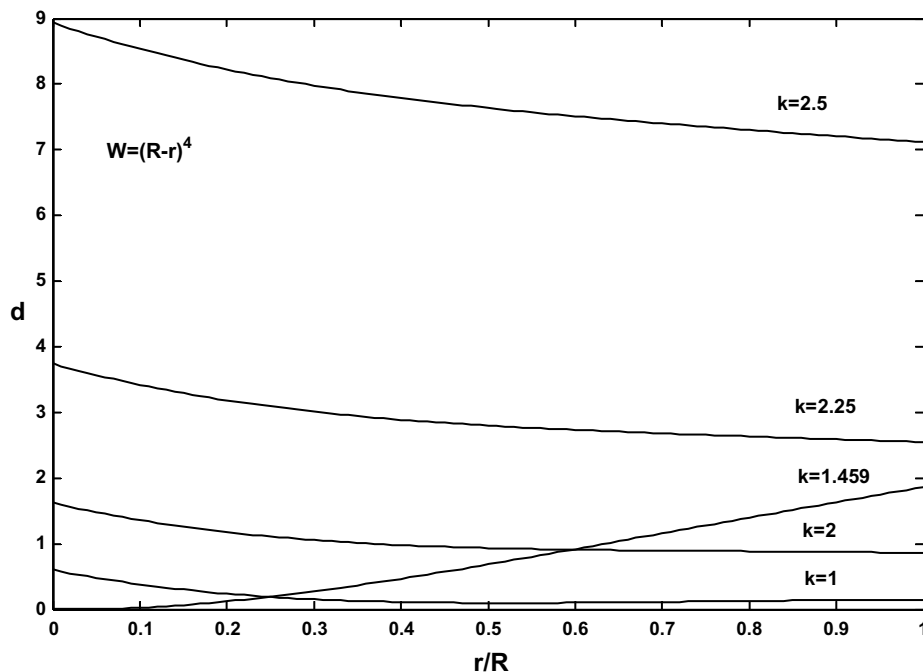


Fig. 8. Variation of d vs. non-dimensional radial coordinate r/R for various values of k , and $v_r = 0.35$.

polar orthotropic plate, at $\nu_r = 0.35$, it turns out that the minimum value of k for which the function D_r is non-negative is, 1.459. Fig. 8 shows $d(r)$ for $k = 1.459; 1.75; 2; 2.5$. Note that for $k = 3$ the solution in Eq. (97) is not valid because the denominator contains the factor $k^2 - 9$. Likewise, for any combination of k and ν_r for which either of the following equalities

$$k^2 - 3k^2\nu_r - 18 = 0; \quad k^2 - 2k^2\nu_r - 15 = 0; \quad k^2 - k^2\nu_r - 12 = 0 \quad (98)$$

holds, there is no solution in the form discussed.

Consider now the alternative quartic mode shape:

$$W(r) = R^4 - 4r^3R + 3r^4 \quad (99)$$

Note that this function was utilized in (Elishakoff, 1987) as a trial function in the context of the approximate, Rayleigh–Ritz method. Herein we show that Eq. (92) can also serve as exact mode shape. The substitution of Eqs. (99), (11) and (12) into Eq. (9) leads to the equation:

$$\sum_{j=0}^5 F_j r^j = 0 \quad (100)$$

where

$$F_0 = 24k^2b_0\nu_rR - 48b_0R - 24b_0\nu_\theta R + 12k^2b_0R \quad (101)$$

$$F_1 = -72k^2b_0\nu_r - 144b_1R + 72b_0\nu_\theta - 24k^2b_0 + 25k^2b_1R + 216b_0 - 72b_1\nu_\theta R + 48k^2b_1\nu_rR - \rho h\omega^2R^4 \quad (102)$$

$$F_2 = 144b_1\nu_\theta - 144b_2\nu_\theta R + 432b_1 + 36k^2b_2R - 108k^2b_1\nu_r + 72k^2Rb_2\nu_r - 36b_1k^2 - 288b_2R \quad (103)$$

$$F_3 = -240b_0b_3R - 144k^2b_2\nu_r + 720b_2 - 48k^2b_2 - 480b_3R + 240b_2\nu_\theta + 48k^2Rb_3 + 96k^2b_3\nu_rR \quad (104)$$

$$F_4 = -180k^2b_3\nu_r - 720b_4R - 360b_4\nu_\theta R + 1080b_3 + 360b_3\nu_\theta + 60b_4k^2R - 60k^2b_3 + 120k^2b_4\nu_rR + 4\rho h\omega^2R \quad (105)$$

$$F_5 = -216k^2b_4\nu_r - 72k^2b_4 + 504b_4\nu_\theta + 1512b_4 - 3\rho h\omega^2 \quad (106)$$

From Eq. (106) we get the expression of the natural frequency squared ω^2 with respect of the coefficient b_4 :

$$\omega^2 = 24(21 + 7\nu_\theta - k^2 - 3k^2\nu_r)b_4/\rho h \quad (107)$$

The coefficients of flexural rigidity are as follows:

$$b_0 = \frac{Q_9}{5(k^2 - 6\nu_\theta + 3k^2\nu_r - 18)(k^2 - 15 - 5\nu_\theta + 3k^2\nu_r)(k^2 + 4k^2\nu_r - 4\nu_\theta - 12)(k^2 - 3\nu_\theta - 9 + 3k^2\nu_r)} R^4 b_4 \quad (108)$$

$$b_1 = -\frac{(3k^2 - 26\nu_\theta + 14k^2\nu_r - 08)(k^2 - 10 + 2k^2\nu_r - 5\nu_\theta)(k^2 + 2k^2\nu_r - 4\nu_\theta - 8)}{5(k^2 - 6\nu_\theta + 3k^2\nu_r - 18)(k^2 - 15 - 5\nu_\theta + 3k^2\nu_r)(k^2 + 4k^2\nu_r - 4\nu_\theta - 12)} R^3 b_4 \quad (109)$$

$$b_2 = -\frac{(3k^2 - 26\nu_\theta + 14k^2\nu_r - 08)(k^2 - 10 + 2k^2\nu_r - 5\nu_\theta)}{5(k^2 - 6\nu_\theta + 3k^2\nu_r - 18)(k^2 - 15 - 5\nu_\theta + 3k^2\nu_r)} R^2 b_4 \quad (110)$$

$$b_3 = -\frac{1}{5} \frac{(3k^2 + 14k^2\nu_r - 26\nu_\theta - 108)}{(k^2 - 6\nu_\theta + 3k^2\nu_r - 18)} R b_4 \quad (111)$$

where

$$\begin{aligned} Q_9 = & (-2090k^6\nu_r^3\nu_\theta + 5491k^4\nu_r^2\nu_\theta^2 + 3182k^4\nu_r\nu_\theta^2 - 522k^6\nu_\theta\nu_r + 20780k^4\nu_\theta\nu_r - 64k^2 - 211062k^2\nu_r + 34201k^4\nu_r \\ & - 58778k^2\nu_\theta + 4899k^4 + 169200\nu_\theta^2 + 56863k^4\nu_r^2 + 35386k^4\nu_r^2\nu_\theta - 197926k^2\nu_r\nu_\theta - 1698k^6\nu_r + 363360\nu_\theta \\ & + 34560\nu_\theta^3 - 6702k^6\nu_r^3 - 5982k^6\nu_r^2 - 61162k^2\nu_r\nu_\theta^2 - 1766k^2\nu_\theta^2 288360) \end{aligned} \quad (112)$$

With $\nu_\theta = k^2\nu_r$ the final expression of the flexural rigidity reads, in conjunction with Eq. (107)

$$\begin{aligned} D_r(r) = & \left[\frac{1}{5(k^2 - 3k^2\nu_r - 18)(k^2 - 15 - 2k^2\nu_r)(k^2 - 12 - k^2\nu_r)(k^2 - 9)} (152298k^2\nu_r + 28137k^4\nu_r^2 - 64386k^4\nu_r + 4899k^4 \right. \\ & - 2868k^6\nu_r^2 + 2082k^6\nu_r^3 + 1188k^6\nu_r - 20k^8\nu_r + 70k^8\nu_r^2 - 100k^8\nu_r^3 - 24576k^4\nu_r + 288360 - 150k^6 + 2k^8 \\ & + 48k^8\nu_r^4) R^4 - \frac{(3k^2 - 12k^2\nu_r - 108)(k^2 - 3k^2\nu_r - 10)(k^2 - 2k^2\nu_r - 8)}{5(k^2 - 3k^2\nu_r - 18)(k^2 - 15 - 2k^2\nu_r)(k^2 - 12 - k^2\nu_r)} R^3 r \\ & \left. - \frac{1}{5} \frac{(3k^2 - 12k^2\nu_r - 108)(k^2 - 3k^2\nu_r - 10)}{(k^2 - 3k^2\nu_r - 18)(k^2 - 15 - 2k^2\nu_r)} R^2 r^2 - \frac{1}{5} \frac{(3k^2 - 12k^2\nu_r - 108)}{(k^2 - 3k^2\nu_r - 18)} R r^3 + r^4 \right] \frac{\Omega^2}{A_6}, \\ A_6 = & 24(21 + 7\nu_\theta - k^2 - 3k^2\nu_r)/\rho h \end{aligned} \quad (113)$$

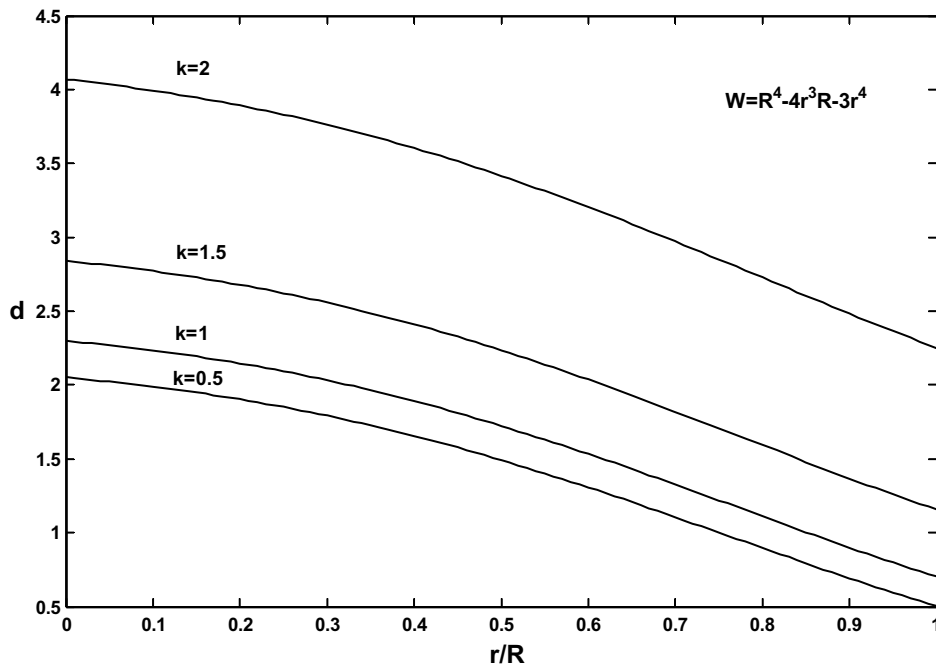


Fig. 9. Variation of d vs. non-dimensional radial coordinate r/R for various values of k , and $\nu_r = 0.35$.

Fig. 9 presents the flexural rigidity for various values of k . It must be noted that the mode shape in Eq. (99) is proportional to the static displacement of the polar orthotropic circular plate when is specified at $k = 2$. Indeed, in his classic book¹² Lekhnitskii derives the formula for the displacement of the polar orthotropic circular plate under uniform loading q_0

$$w = \frac{q_0 R^4}{8(9 - k^2)(1 + k)D_r} \left[3 - k - 4 \left(\frac{r}{R} \right)^{1+k} + (1 + k) \left(\frac{r}{R} \right)^4 \right] \quad (114)$$

For $k = 2$ the expression in brackets reduces to Eq. (92). Eq. (113) that is obtained with Eq. (99) as the mode shape, however is not restricted to the value $k = 2$. It is applicable for any value of k for which the denominator in Eq. (113) does not vanish. The following unintuitive conclusion is reached: Lekhnitskii's static displacement function, when it is specified at $k = 2$, can serve as an exact mode shape of vibrating polar orthotropic plate both for $k = 2$ and $k \neq 2$.

7. Conclusion

Apparently for the first time, six closed-form solutions have been derived for the polar orthotropic plate that is clamped at its circumference. The appropriate mode shapes and the natural frequency expressions are summarized in Table 1. It must be stressed, that the expressions for the fundamental natural frequency corresponding to two cubic mode shapes coincide with each other. However, the associated circular plates possess differing flexural rigidities, given in Eqs. (58) and (73), respectively. Likewise, the natural frequencies associated with discussed three quartic mode shapes $(R^2 - r^2)^2$, $(R - r)^4$ and $R^4 - 4r^3R + 3r^4$ coincide. The corresponding flexural rigidities are given, respectively, in Eqs. (26), (90) and (106), and differ from each other, as they should be.

This study demonstrates that the analytical semi-inverse method provides a possibility for straightforward vibration tailoring. Such a tailoring recently became extremely important in various fields of engineering. It appears remarkable that the

Table 1
Mode shapes, associated natural frequencies and flexural rigidities of heterogeneous circular plates

Cases	Mode shapes	Fundamental natural frequencies	Equation number where flexural rigidity is given
1	$W(r) = (R^2 - r^2)^2$	$\omega^2 = 24(21 + 7\nu_0 - k^2 - 3k^2\nu_r)b_4/\rho h$	26
2	$W(r) = (R - r)^2$	$\omega^2 = 8(5 + 5\nu_0 - k^2 - k^2\nu_r)b_4/\rho h$	46
3	$W(r) = (R - r)^3$	$\omega^2 = 15(12 + 6\nu_0 - k^2 - 2k^2\nu_r)b_4/\rho h$	65
4	$W(r) = R^3 - 3r^2R + 2r^3$	$\omega^2 = 15(12 + 6\nu_0 - k^2 - 2k^2\nu_r)b_4/\rho h$	80
5	$W(r) = (R - r)^4$	$\omega^2 = 24(21 + 7\nu_0 - k^2 - 3k^2\nu_r)b_4/\rho h$	97
6	$W(r) = R^4 - 4r^3R + 3r^4$	$\omega^2 = 24(21 + 7\nu_0 - k^2 - 3k^2\nu_r)b_4/\rho h$	113

vibrational tailoring problem is resolved here in an analytical closed-form manner. Thus, the problem that is posed and solved herein, can serve as a benchmark case for various vibration tailoring problems which usually have to be attacked numerically.

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